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GAMES OF PREDICTION OF PERIODIC SEQUENCES (U)

by

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In this paper several infinite two-person games are studied, all having the following common structure: Player 1 (Emitter) produces a binary periodic sequence. Player 2 (Predictor) observes some initial segment of this sequence and then tries to predict the next digit. The payoff to Emitter is zero if the prediction is a correct one. The games differ in additional assumptions--those are in particular: (1) Predictor required to make his prediction after observing a prescribed number of digits of the sequence. (2) Predictor allowed to observe any number of digits but earning a decreasing amount for each correct prediction as the number increases. (3) The period of the emitted sequence being chosen by random from some fixed distribution. (4) Emitter allowed to choose the period but being paid a decreasing amount for incorrect prediction as the period increases. Combining these assumptions two zero-sum and two nonzero-sum games are obtained. It is shown that all these games possess a solution, some are at least partially solved and their further properties investigated.

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1. INTRODUCTION.

In this paper we study certain types of infinite games which belong to the class of discrete emission-prediction games, also known as discrete games of aiming and evasion. An emission-prediction game is a two-person game, where the first player, to be named Emitter, produces a sequence of elements from some fixed finite set A. The second player, to be named Predictor, is allowed to observe the sequence for some time and is then required to make a prediction of some kind about the future behavior of the sequence. The general Predictor's goal is to make a correct prediction, while Emitter wants to avoid this.

Emission-prediction games may serve as models for a variety of conflict situations. For instance, Emitter could be identified as an attacker (bomber, submarine, guided missile, guerilla unit) performing a series of evasive maneuvers, which, in the discrete time, represent the sequence being emitted. Predictor is trying to destroy the attacker. Since, in general, the attacker can be intercepted only after one or more time units have elapsed from the last observed maneuver (or position), Predictor must be able to correctly predict future maneuvers (positions) of the attacker.

To our best knowledge the study of emission-prediction games was initiated by Isaacs (the "bomber and battleship" problem and have since been investigated by Dubins [2], Isaacs and Karlin [5],

Isaacs [4], Karlin [6], Blackwell [1], Ferguson [3] and Matula [7]. The assumptions made by these authors can be summarized as follows:

(1) No restriction is placed upon the emitted sequence.

(2) Predictor is allowed to observe the sequence as long as he wishes before he decides to make a prediction concerning several subsequent terms of the sequence.

(3) The game is a zero-sum game with the payoff depending only on the discrepancy between the prediction and the actual values.

The emission-prediction games investigated in this paper differ mainly in one crucial aspect; we assume that the emitted sequence is a periodic one. Apart from this being of interest per se we have been motivated by the following idea. Suppose that the attacker is a simple automatic device with built-in preprogrammed ability to perform evasive maneuvers. The trajectory of this device will then follow a periodic pattern with some period depending upon the complexity of the program and unknown to Predictor--the defender. Even in situations where the attacker is controlled by a human operator periodicity may very well serve as a first approximation. We believe that most humans do exhibit a kind of cyclic pattern when asked to perform a series of evasive, i.e. unpredictable maneuvers.

Assuming periodicity of the emitted sequence, however, compels us to place some additional restrictions on Emitter and/or

Predictor since otherwise the game would not possess a solution. The additional restrictions considered in this paper are: the period is chosen by an independent chance mechanism, bounds on the length of observation intervals, and penalties for long periods and lengthy observation. As for the type of prediction required we limited outselves to the simplest case of predicting the next term of the sequence. (With the periodicity assumptions this is by no means trivial as it would be without it.) For the sake of simplicity of notation we consider binary sequences only; most of the results of this paper extend easily to the m-ary case.

Another possible application of the model may be that of an optimum jamming strategy for a missile defense system. Suppose that a ship is being attacked by a missile equipped with a target search radar. To avoid detection the search radar is not active all the time but rather is constantly being switched on and off in some programmed pattern. The intensity of illumination of the target by the search radar then follows a binary periodic sequence (in a discrete time). The defender of the target (ship) wants to jam the search radar. The jamming device, however, must not be transmitting all the time since the missile could then home on the transmitter and hence the target. Thus, the defender faces the problem of predicting the next mode of operation of the search radar after observing the illumination pattern for some limited time, i.e., predicting next term of the binary sequence.

2. PRELIMINARIES.

Throughout this paper the symbol A will denote the two-element set $A = \{0,1\}$ and the symbol Ω with generic elements ω will denote the set of all periodic sequences of zeros and ones. We say that a binary sequence

$$\omega = (\omega_1,\omega_2,\ldots) \; ; \quad \omega_n \in A; \quad n = 1,2,\ldots;$$

is periodic with the period $\pi(\omega) = t; \quad t = 1,2,\ldots,$ if

$$\omega_{n+kt} = \omega_n \quad \text{for every} \quad n = 1,2,\ldots; \quad k = 1,2,\ldots \; .$$

Obviously, each periodic sequence $\omega \in \Omega$ has only one period, and we denote

$$\Omega_t = \{\omega \in \Omega : \pi(\omega) = t\}$$

so that Ω is a disjoint union of $\Omega_1,\Omega_2,\ldots$.

If $\omega \in \Omega$ and $n = 1,2,\ldots$ then ω_n will denote the n-th term and $\underline{\omega}|^n$ the ordered n-tuple of the first n terms of the sequence ω.

Let A^n be the set of all ordered n-tuples of zeros and ones. If $a = (\alpha_1,\ldots,\alpha_n) \in A^n$ and $\beta \in A$ then $a\beta$ is the ordered $(n+1)$-tuple $(\alpha_1,\ldots,\alpha_n,\beta)$ and $(a)^\infty$ is the periodic sequence

$\omega = (\alpha_1,\ldots,\alpha_n,\alpha_1,\ldots,\alpha_n,\ldots)$. Clearly, $\pi((a)^\infty) \leq n$.

For every $a \in A^n$ and $t = 1,2,\ldots$ we define the function

$$I_t(a) = \begin{cases} 1 & \text{if there is } \omega \in \Omega_t \text{ such that } \omega = (a)^\infty, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and the function

$$\lambda(a) = \min\{t = 1,2,\ldots : I_t(a) = 1\}. \tag{2}$$

Thus $I_t(a)$ serves as an indicator of "potential periods" of a and $\lambda(a)$ is the "smallest potential period" of a.

Later we will need the following lemma,

Lemma 2.1: For every $a \in A^n$ there exists $\alpha \in A$ such that $\lambda(a\alpha) > \left[\frac{n}{2}\right] + 1$, where $[x]$ denotes the integral part of the number x.

Proof: For any n-tuple $a = (\alpha_1,\ldots,\alpha_n)$ denote

$$T(a) = \{t = 1,\ldots,n-1 : I_t(a) = 1\} \quad \text{if} \quad n > 1$$

and $T(a) = \emptyset$ if $n = 1$. Notice first that

$$T(a0) \cap T(a1) = \emptyset$$

and (3)

$$T(a0) \cup T(a1) = T(a) \cup \{n\}.$$

This follows by realizing that if $t \in T(a)$ then $a = (\alpha_1,\ldots,\alpha_t; \alpha_1,\ldots,\alpha_t;\ldots;\alpha_1,\ldots,\alpha_i)$, $1 \le i \le t$ so that $t \in T(a\alpha_{i+1})$

if $i < t$ and $t \in T(a\alpha_1)$ if $i = t$. Conversely, if $t \in T(a\alpha_{i+1})$ and $t < n$ then $t \in T(a)$ and clearly $t \notin T(a\alpha)$ for $\alpha \neq \alpha_{i+1}$. Finally, $n \in T(a\alpha)$ if and only if $\alpha \neq \alpha_1$.

Let now $a = (\alpha_1,\ldots,\alpha_n)$ and assume $n > 2$ since the lemma is trivial for $n = 1,2$. If $T(a) = \emptyset$ then by (3) $T(a0) \cup T(a1) = \{n\}$ so that $\lambda(a\alpha) \geq n$ for any $\alpha \in A$.

If $T(a) \neq \emptyset$ then there is $b = (\beta_1,\ldots,\beta_k)$ and $c = (\gamma_1,\ldots,\gamma_\ell)$ such that

$$a = \underbrace{(bc\ bc\ \ldots\ bc}_{m \text{ times}}\ b) \quad \text{and} \quad T(bc) = \emptyset.$$

If $T(b) = \emptyset$ then $T(a) = \{k + \ell\}$ and by (3) $\lambda(a\alpha) \geq n$ for some $\alpha \in A$.

If $T(b) \neq \emptyset$ then there is $e = (\varepsilon_1,\ldots,\varepsilon_r)$ and $d = (\delta_1,\ldots,\delta_s)$ such that

$$b = (ed\ ed\ \ldots\ ed\ e) \quad \text{and} \quad T(ed) = \emptyset.$$

Hence taking $\alpha \neq \varepsilon_1 = \beta_1$ we have

$$\lambda(a\alpha) \geq m(k+\ell) + r + s \geq k + \ell + 2 = n - k + 2$$

and since $2k \leq n - 1$ we obtain

$$\lambda(a\alpha) \geq \frac{n-1}{2} + 2 > \left[\frac{n}{2}\right] + 1. \quad \text{The lemma is proved.}$$

3. ZERO-SUM GAMES WITH RANDOM PERIOD.

In this section we consider the case where

(1) the period t of Emitter's sequence is chosen randomly according to some distribution ν known to both players,

(2) Predictor is allowed to observe the first n digits of the Emitter's sequence and tries to predict the next $(n+1)$st digit,

(3) the payoff to Emitter is zero if the prediction is correct and one otherwise.

In other words, the expected payoff is the probability of wrong prediction, which the Emitter is trying to maximize and the Predictor to minimize.

We will denote the resulting zero-sum game by $G(\nu,n)$, where $\nu(t)$, $t = 1,2,\ldots$ is the period distribution,

$$\nu(t) \geq 0, \quad t = 1,2,\ldots ; \qquad \sum_{t=1}^{\infty} \nu(t) = 1$$

and $n = 1,2,\ldots$ is the length of the sequence observed by the Predictor.

Next we describe Emitter's and Predictor's pure strategies.

To specify a pure strategy Emitter must choose for every $t = 1,2,\ldots$ a binary sequence with the period t. Emitter's pure strategies s are therefore sequences

$$s = \{s_t, \ t = 1,2,\ldots\}, \quad \text{where} \quad s_t \in \Omega_t .$$

The set S of all Emitter's pure strategies is then an infinite Cartesian product $S = \Omega_1 \times \Omega_2 \times \ldots$ of finite sets and is therefore uncountable.

Predictor's pure strategies ϕ are simpler to describe. Each ϕ is a mapping of A^n into $A = \{0,1\}$. For each $a \in A^n$, $\phi(a)$ is the prediction of the $(n+1)$st term of the Emitter's sequence if the first n terms observed are the n-tuple a. The set Φ of all Predictor's pure strategies is therefore a finite set of 2^{2^n} elements.

The payoff function $W(s,\phi)$ is then given by

$$W(s,\phi) = \sum_{t=1}^{\infty} \nu(t)\, W_t(s,\phi) \; ; \quad s \in S, \quad \phi \in \Phi,$$

where for $t = 1,2,\ldots$ and $\omega = s_t$,

$$W_t(s,\phi) = \begin{cases} 1 & \text{if } \phi(\underline{\omega}|^n) \neq \omega_{n+1}, \\ 0 & \text{if } \phi(\underline{\omega}|^n) = \omega_{n+1}. \end{cases}$$

It is easy to see that $W_t(s,\phi)$ is nothing but conditional probability of wrong prediction given that the period is t and hence $W(s,\phi)$ is the unconditional error probability.

The game $G(\nu,n)$ is now formally defined by the triplet (S,Φ,W). This is an infinite zero-sum game, however, since the set Φ is finite and the payoff function is nonnegative and bounded by one we have by the well-known theorem of Wald [8] the following proposition.

Proposition 3.1: The game $G(\nu,n)$ has a value and Predictor has an optimal strategy.

Let us turn our attention to the existence of Emitter's optimal strategy. By another Wald's theorem ([8], Theorem 2.20) an optimal strategy exists if the corresponding space of pure strategies is totally bounded with respect to the intrinsic metric ([8], sec. 2.1). Now the space S of Emitter's pure strategies is totally bounded in the above sense if for every $\varepsilon > 0$ there exists a finite subset S_ε of S such that for every $s \in S$ there is an $r \in S_\varepsilon$ for which

$$\max_{\phi\in\Phi} \left| W(s,\phi) - W(r,\phi) \right| < \varepsilon.$$

It is easy to see that this condition is satisfied for the game $G(\nu,n)$; take an integer T such that $\sum_{t>T} \nu(t) < \varepsilon$ and define S_ε to be the set of all pure strategies $s = \{s_t,\ t = 1,2,\ldots\}$ with s_t fixed for $t > T$. Thus we have proved the following proposition.

Proposition 3.2: In the game $G(\nu,n)$ Emitter has also an optimal strategy.

In general, optimal strategies will be mixed strategies, that is probability distributions on the corresponding spaces of pure strategies.

Let X be the space of Emitter's mixed strategies. Since S is a product space a mixed strategy $x \in X$ will be a product probability measure

$$x = x_1 \times x_2 \times \dots ,$$

where x_t is a probability distribution on the finite set Ω_t. In other words, $x \in X$ is completely specified by a sequence of probability distributions x_t; $t = 1,2,\dots$. The payoff function W then extends onto $X \times \Phi$ by

$$W(x,\phi) = \int_S W(s,\phi)dx(s) = \int_S \sum_{t=1}^{\infty} \nu(t)W_t(s,\phi)dx(s)$$

$$= \sum_{t=1}^{\infty} \nu(t) \int_S W_t(s,\phi)dx(s) = \sum_{t=1}^{\infty} \nu(t) \sum_{s_t \in \Omega_t} W_t(s,\phi)x_t(s_t).$$

Predictor's mixed strategies $y \in Y$ are again probability distributions on the finite set Φ. Thus for every such y and $s \in S$ we can define

$$W(s,y) = \sum_{\phi \in \Phi} W(s,\phi)y(\phi) = \sum_{t=1}^{\infty} \nu(t) \sum_{\phi \in \Phi} W_t(s,\phi)y(\phi).$$

Let $y \in Y$ and let for every $a \in A^n$, $\alpha \in \{0,1\}$ $f_\alpha(a) = \sum_{\{\phi:\phi(a)=\alpha\}} y(\phi)$

Clearly, $f_\alpha(a)$ is the probability of predicting α if the n-tuple a has been observed and if Predictor uses the mixed strategy y. Thus $f = (f_0,f_1)$ could be called Predictor's behavioral strategy and we have for $\omega = s_t$

$$W_t(s,f) = \sum_{\phi\in\Phi} W_t(s_1\phi)y(\phi) = \begin{cases} f_0(\underline{\omega}|^n) & \text{if } \omega_{n+1} = 1, \\ f_1(\underline{\omega}|^n) & \text{if } \omega_{n+1} = 0. \end{cases} \tag{4}$$

Conversely, if $f = (f_0,f_1)$ where f maps A^n into $[0,1]$ and $f_0(a) + f_1(a) = 1$ for all $a \in A^n$ then

$$y(\phi) = \sum_{\{a\in A^n:\phi(a)=0\}} f_0(a) + \sum_{\{a\in A^n:\phi(a)=1\}} f_1(a), \qquad \phi \in \Phi,$$

is a mixed strategy and (4) holds. Hence we can work with the set F of Predictor's behavioral strategies instead of with the set Y.

Proposition 3.3: The value v of $G(\nu,n)$ is given by

$$v = \min_{f\in F} \sum_{t=1}^{\infty} \nu(t) \max_{a\in A^n} \max\{I_t(a0)f_1(a), I_t(a1)f_0(a)\} \tag{5}$$

and f^* is Predictor's optimal strategy if and only if it minimizes the series above.

Proof: Since $G(\nu,n)$ has a value we must have $v = \min_{f\in F} \sup_{s\in S} W(s,f)$. Now

$$\sup_{s\in S} W(s,f) = \sum_{t=1}^{\infty} \nu(t) \max_{s_t\in\Omega_t} W_t(s,f),$$

and by (4)

$$\max_{s_t\in\Omega_t} W_t(s,f) = \max\{f_{1-\alpha}(a)\},$$

where the latter maximum is taken over the set of all (n+1)-tuples $a\alpha \in A^{n+1}$ for which $\underline{\omega}|^{n+1} = a\alpha$ for some $\omega \in \Omega_t$, i.e. for which $I_t(a\alpha) = 1$. Hence

$$\max_{s_t \in \Omega_t} W_t(s,f) = \max_{a \in A^n} \max\{I_t(a0)f_1(a), I_t(a1)f_0(a)\}$$

and the second assertion follows from the existence of Predictor's optimal strategy.

Setting $f(a) = (\frac{1}{2},\frac{1}{2})$ for each $a \in A^n$ in (5) we obtain

Corollary: $$0 \leq v \leq \frac{1}{2}$$

Proposition 3.4: There exists Predictor's optimal strategy f^* such that for each $a \in A^n$ either $f^*(a) = (1,0)$ or $f^*(a) = (0,1)$ or $f^*(a) = (\frac{1}{2},\frac{1}{2})$.

Proof: Let for every $t = 1,\ldots,n+2$ and $a \in A^n$

$$B(t,a) = \{\underline{f}_0 : f_0(a) = \max_{a \in A^n} \max\{I_t(a0)f_1(a), I_t(a1)f_0(a)\}\},$$

where $\underline{f}_0$ denotes the vector with 2^n components $f_0(a)$. Clearly, $B(t,a)$ is a convex polyhedron contained in the 2^n-dimensional unit hypercube. Furthermore, the vertices of $B(t,a)$ are vectors with components 0 or 1 or 1/2 only.

Let $u = (a_1,\ldots,a_{n+2})$, $a_i \in A^n$ and let

$$L_u(\underline{f}_0) = \sum_{t=1}^{n+1} \nu(t) f_0(a_t) + f_0(a_{n+2}) \sum_{t>n+1} \nu(t).$$

Now L_u is a linear function of $\underline{f}_0$ and hence its minimum over the convex polyhedron

$$B_u = \bigcap_{t=1}^{n+2} B(t,a_t)$$

is attained at one of the vertices of B_u. Denoting $\underline{f}_0^u$ the vertex at which the minimum is attained we see that for each $a \in A^n$

$$f_0^u(a) = 0 \quad \text{or} \quad 1/2 \quad \text{or} \quad 1. \tag{6}$$

Finally, since $I_t(a\alpha) = 1$ for $t > n + 1$,

$$\min_f \sum_{t=1} \nu(t) \max_{a\in A^n} \max\{I_t(a0)f_1(a), I_t(a1)f_0(a)\}$$

$$= \min_{u\in A^{n(n+2)}} \min_{\underline{f}_0 \in B_u} L_u(\underline{f}_0) = \min_{u\in A^{n(n+2)}} L_u(\underline{f}_0^u)$$

$$= L_{u^*}(\underline{f}_0^{u^*}), \quad \text{where } u^* \text{ minimizes } L_u(\underline{f}_0^u).$$

It follows from Proposition 3 that $f^{u^*} = (f_0^{u^*}, 1-f_0^{u^*})$ is an optimal strategy which together with (6) terminates the proof.

Proposition 3.5: Let $\sum_{t>n+1} \nu(t) > \frac{1}{2}$. Then $v = \frac{1}{2}$ and $f(a) = (\frac{1}{2},\frac{1}{2})$, $a \in A^n$ is the unique Predictor's optimal strategy.

Proof: Let f^* be an optimal strategy and assume that for some $a \in A^n$, $\alpha \in \{0,1\}$ $f_\alpha^*(a) \neq \frac{1}{2}$. It follows from the proof of

Proposition 3.4 that there must be another optimal strategy with $f^*_\alpha(a) = 0$ or 1. Hence assume that $f^*_\alpha(a) = 1$ (otherwise take $f_{1-\alpha}(a)$). Then

$$\sum_{t=1}^{\infty} \nu(t) \max_{a\in A^n} \max\{I_t(a0)f^*_1(a), I_t(a1)f^*_0(a)\}$$

$$\geq \max_{a\in A^n} \max\{f^*_1(a), f^*_0(a)\} \sum_{t>n+1} \nu(t) > \frac{1}{2}$$

so that $v > \frac{1}{2}$, which is a contradiction. Hence $f^*_\alpha(a) = \frac{1}{2}$ for all $a \in A^n$, $\alpha \in \{0,1\}$. For this f^*

$$v = \sum_{t=1}^{\infty} \nu(t) \frac{1}{2} \max_{a\in A^n} \max\{I_t(a0), I_t(a1)\} = \frac{1}{2}$$

since for every $t = 1,2,\ldots$ there exists $a \in A^n$ and $\alpha \in \{0,1\}$ such that $I_t(a\ ) = 1$.

Proposition 3.6: Let the distribution ν be such that

$$\nu([\tfrac{n}{2}] + 1) > \sum_{t>[\frac{n}{2}]+1} \nu(t), \tag{7}$$

where $[\frac{n}{2}]$ is the integral part of $\frac{n}{2}$. Then

$$v = \sum_{t>[\frac{n}{2}]+1} \nu(t) \tag{8}$$

and the Predictor's behavioral strategy f^* defined by

$$\lambda(a\alpha) \leq [\tfrac{n}{2}] + 1 \Rightarrow f^*_\alpha(a) = 0 \tag{9}$$

is an optimal strategy.

Proof: Let us assume first that n is odd so that $[\frac{n}{2}] + 1 = \frac{n+1}{2}$. Let $e \in A^n$ be the n-tuple which has 1 at the $(\frac{n+1}{2})$th place and zeros elsewhere. Clearly

$$I_t(e0) = 1 \text{ if an only if } t > \frac{n+1}{2},$$

and

$$I_t(e1) = 1 \text{ if and only if either } t = \frac{n+1}{2} \text{ or } t > n+1.$$

Hence

$$\sum_{t=1}^{\infty} \nu(t) \max_{a\in A^n} \max\{I_t(a0)f_1(a), I_t(a1)f_0(a)\}$$

$$\geq \sum_{t=1}^{\infty} \nu(t) \max\{I_t(e0)f_1(e), I_t(e1)f_0(e)\}$$

$$= \nu\left(\tfrac{n+1}{2}\right) f_0(e) + f_1(e) \sum_{\frac{n+1}{2}<t\leq n+1} \nu(t) + \max\{f_1(e), f_0(e)\} \sum_{t>n+1} \nu(t)$$

$\geq \sum_{t>\frac{n+1}{2}} \nu(t)$ by the hypothesis. Thus $v \geq \sum_{t>\frac{n+1}{2}} \nu(t)$.

Next let f^* be the strategy (9). Then by Lemma 2.1

$$v \le \sum_{t=1}^{\infty} \nu(t) \max_{a\in A^n} \max\{I_t(a0)f_1^*(a), I_t(a1)f_0^*(a)\}$$

$$= \sum_{t>\frac{n+1}{2}} \nu(t) \max_{a\in A^n} \max\{I_t(a0)f_1^*(a), I_t(a1)f_0^*(a)\}$$

$$= \sum_{t>\frac{n+1}{2}} \nu(t).$$

Hence $v = \sum_{t>\frac{n+1}{2}} \nu(t)$ and f^* is an optimal strategy.

If n is even we begin with e having 1 at the $(\frac{n}{2}+1)$st place. The rest follows verbatim.

If the hypothesis of the above proposition is not satisfied the game $G(\nu,n)$ is very difficult to solve. As seen from Proposition 3.3, this amounts to solving the nonlinear programming problem:

$$\text{"minimize} \quad \sum_{t=1}^{\infty} \nu(t) \max_{a\in A^n} \max\{I_t(a0)f_1(a), I_t(a1)f_0(a)\} \qquad (10)$$

$$\text{subject to} \quad f \in F."$$

It is true that it can be solved by breaking it into a number of linear programming problems as we did in the proof of Proposition 3.4 but the size of this task is still formidable. A slight simplification, however, can be obtained by considering truncated versions of $G(\nu,n)$.

Let $T = 1,2,\ldots,$ let $G_T(\nu,n)$ be the game (S,Φ,W^T) obtained from $G(\nu,n)$ by truncating the distribution ν at T, that is setting $\nu(t) = 0$ for $t > T$ and normalizing to one.

Proposition 3.7: Let v^T, f^T and x^T be the value and optimal strategies respectively of the truncated game $G_T(\nu,n)$. Then, as $T \to \infty$,

$$v^T \to v, \quad f^T \to f^* \quad \text{and} \quad x^T \to x^*,$$

where v, f^* and x^* are the value and optimal strategies of $G(\nu,n)$.

Proof: Since ν is a probability distribution and

$$|W^T(x,f) - W(x,f)| \leq \sum_{t>T} \nu(t)$$

$W^T(x,f)$ converges to $W(x,f)$ uniformly in both f and x, which implies the statement.

To the end of this section we present a solution of a small game $G(\nu,n)$.

Example: Let $n = 5$, $\nu(t) > 0$ for $t = 1,\ldots,5$ and $\nu(t) = 0$ for $t > 5$. To compute the right-hand side of (5) we need the values of $I_t(a\alpha)$ for $t = 1,\ldots,5$, $a \in A^5$, $\alpha = 0,1$. These are given in the first two columns of Table 1. Substituting into (10) we obtain

$$\nu(1)f_1(a_1) + \nu(2)f_0(a_{11}) + \nu(3)\max\{f_0(a_5),f_1(a_{10}),f_0(a_{14})\}$$

$$+ \nu(4)\max\{f_1(a_3),f_1(a_5),f_1(a_7),f_0(a_9),f_0(a_{13}),f_0(a_{15})\}$$

$$+ \nu(5)\max\{f_1(a_i);\ i = 2,\ldots,16\}$$

$$+ \nu(1)f_0(a_{32}) + \nu(2)f_1(a_{22}) + \nu(3)\max\{f_1(a_{28}),f_0(a_{23}),f_1(a_{19})\}$$

$$+ \nu(4)\max\{f_0(a_{30}),f_0(a_{28}),f_0(a_{26}),f_1(a_{24}),f_1(a_{20}),f_1(a_{18})\}$$

$$+ \nu(5)\max\{f_0(a_i):\ i = 17,\ldots,31\}. \qquad (11)$$

Looking at the expression (11) as a function of 64 nonnegative variables $f_j(a_i)$ constrained to satisfy $f_0(a_i) + f_1(a_i) = 1$ we see that the only variables that make the minimization difficult are those for which both $f_0(a_i)$ and $f_1(a_i)$ appear in (11). Hence it is easy to conclude that the minimum of (11), the value v of the game, is equal to the minimum of

$$\nu(2)u_1 + \nu(3)u_2 + \nu(4)\max\{u_3,1-u_2\} + \nu(4)\max\{1-u_1,1-u_2,1-u_3\} \qquad (12)$$

over all u_1,u_2,u_3 satisfying $0 \le u_j \le 1$, $j = 1,2,3$.
The minimum of (12) is found to be the smallest of the four numbers:
$\nu(4) + \nu(5)$, $\nu(3) + \nu(5)$, $\nu(2) + \nu(3) + \nu(4)$, $\frac{1}{2}(\nu(2)+\nu(3)+\nu(4)+\nu(5))$.
Hence, the value v of the game is:

Case 1: $\nu(4) \le \nu(3)$ and $\nu(4) + \nu(5) \le \nu(2) + \nu(3)$

$$v = \nu(4) + \nu(5)$$

Case 2: $\nu(3) \leq \nu(4)$ and $\nu(3) + \nu(5) \leq \nu(2) + \nu(4)$

$$v = \nu(3) + \nu(5)$$

Case 3: $\nu(2) + \nu(3) + \nu(4) \leq \nu(5)$

$$v = \nu(2) + \nu(3) + \nu(4)$$

Case 4: $\nu(2) + \nu(3) - \nu(5) \leq \nu(4) \leq \nu(3) + \nu(5) - \nu(2)$

$$v = \frac{1}{2}(\nu(2)+\nu(3)+\nu(4)+\nu(5)) = \frac{1}{2}(1-\nu(1))$$

Predictor's optimal strategies for each of the four cases are given in Table 1.

4. ZERO-SUM GAMES WITH PENALTY FOR LONG PERIODS.

We will now relax the restrictions on Emitter and let him choose freely the period of the sequence emitted. We will, however impose a penalty on Emitter for choosing too large periods. The game then proceeds as follows:

(1) Emitter chooses a period $t = 1,2,...$ and a periodic binary sequence with the period t.

(2) Predictor is allowed to observe the first n digits of that sequence and tries to predict the next $(n+1)$st digit.

(3) The payoff to Emitter is zero if the prediction is correct and is equal to a nonnegative constant $c(t)$ if the prediction is wrong and the emitted sequence has period t.

The sequence $c(t)$, $t = 1,2,...$ represents the penalty and we assume that for all $t = 1,2,...$

$$c(t) \geq c(t+1) \geq 0 \tag{13}$$

To avoid the trivial case we assume that $c(t) > 0$ for at least one t.

The resulting zero-sum game is then specified by the sequence (13) and an integer $n = 1,2,...$. We will denote it by $G(c(t),n)$.

The space S of Emitter's pure strategies s now consists of ordered pairs

$$s = (t,s_t) \quad \text{where} \quad t = 1,2,... \quad \text{and} \quad s_t \in \Omega_t.$$

Hence S is countably infinite.

The space Φ of Predictor's pure strategies as same as same as before.

The payoff function $W(s,\phi)$, $s \in S$, $\phi \in \Phi$ is for $s = (t,s_t)$, $s_t = \omega$ given by

$$W(s,\phi) = \begin{cases} c(t) & \text{if } \phi(\underline{\omega}|^n) \neq \omega_{n+1}, \\ 0 & \text{if } \phi(\underline{\omega}|^n) = \omega_{n+1}. \end{cases} \tag{14}$$

Introducing again Predictor's behavioral strategies f we have now

$$W(s,f) = \begin{cases} c(t)f_0(\underline{\omega}|^n) & \text{if } \omega_{n+1} = 1, \\ c(t)f_1(\underline{\omega}|^n) & \text{if } \omega_{n+1} = 0. \end{cases}$$

Since the space Φ is finite and by (13) the payoff function is bounded we have

Proposition 4.1: The game $G(c(t),n)$ has a value and Predictor has an optimal strategy.

This time, however, we can obtain explicit expressions.

Proposition 4.2: The value v of $G(c(t),n)$ is given by

$$v = \max_{a \in A^n} \frac{c(\lambda(a0))c(\lambda(a1))}{c(\lambda(a0)) + c(\lambda(a1))} \tag{15}$$

and the strategy f^* defined for all $a \in A^n$ by

$$f_0^*(a) = \frac{c(\lambda(a0))}{c(\lambda(a0)) + c(\lambda(a1))} \tag{16}$$

is Predictor's optimal strategy.

Remark: Expressions $\frac{0}{0}$ are to be interpreted as 0 in (15) and as any number x, $0 \leq x \leq 1$ in (16).

Proof: Let $f \in F$.

$$\sup_{s\in S} W(s,r) = \sup_{t=1,2,\ldots} \{c(t) \max_{a\in A^n} \max[I_t(a1)f_0(a), I_t(a0)f_1(a)]\}$$

$$= \max_{a\in A^n} \max\{f_0(a) \sup_{t=1,2,\ldots} c(t)I_t(a1), f_1(a) \sup_{t=1,2,\ldots} c(t)I_t(a0)\}.$$

From (13) for $\alpha = 0,1$

$$\sup_{t=1,2,\ldots} c(t)I_t(a\alpha) = c(\lambda(a\alpha)).$$

Hence

$$v = \min_{f\in F} \sup_{s\in S} W(s,r) = \max_{a\in A^n} h(a),$$

where

$$h(a) = \min_{f(a)} \max_{\alpha=0,1} f_\alpha(a)c(\lambda(a\alpha)) . \tag{17}$$

However, $h(a)$ is easily recognized to be the value of the two-by-two matrix game with the matrix

$$H(a) = \begin{bmatrix} 0 & c(\lambda(a0)) \\ c(\lambda(a1)), & 0 \end{bmatrix}$$

so that (15) and (16) hold and the proposition is proved.

Let us now turn our attention to Emitter's strategies. Since the space S is countable, the space X of Emitter's mixed strategies is simply the set of all probability distributions on S. It is, however, more convenient to define a mixed strategy $x \in X$ as the pair

$$x = (\xi, \{\mu_t\}),$$

where $\xi = \xi_1, \xi_2, \ldots$ is a probability distribution on positive integers and μ_t are distributions on finite sets Ω_t. Clearly, if $s = (t, s_t) \in S$ and $x \in X$ then $x(s) = \xi_t \mu_t(s_t)$.

Proposition 4.3: Let $a^* = (\alpha_1^*, \ldots, \alpha_n^*) \in A^n$ be such that

$$h(a^*) = \max_{a \in A^n} h(a), \tag{18}$$

where h is defined by (17), let for $\alpha = 0,1$

$$t_\alpha = \lambda(a^* \alpha),$$

and

$$r_{t\alpha} = (\xi_1^*, \ldots, \xi_{t\alpha}^*) \in \Omega_t$$

Then Emitter's mixed strategy $x^* = (\xi^*, \{\mu_t^*\})$, where

$$\xi_t^* = \begin{cases} C(t_1)[c(t_0) + c(t_1)]^{-1} & \text{if } t = t_0, \\ c(t_0)[c(t_0) + c(t_1)]^{-1} & \text{if } t = t_1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu_t^*(s_t) = \begin{cases} 1 \text{ if } s_t = r_{t_\alpha}, \quad t = t_\alpha, \quad \alpha = 0,1, \\ \text{arbitrary otherwise,} \end{cases}$$

is an optimal strategy.

Proof: Notice first that since by Lemma 2.1

$$\max\{\lambda(a0),\lambda(a1)\} > \left[\frac{n}{2}\right] + 1$$

the $(n+1)$st digits of the two sequences $\omega = r_{t_0}$ and $\omega' = r_{t_1}$ must be different. Let now $\phi \in \Phi$. We have

$$W(x^*,\phi) = \sum_{s\in S} W(s,\phi)x^*(s) = \xi^*_{t_0}\, c(t0)\left|\phi(a^*) - \omega_{n+1}\right|$$

$$+ \xi^*_{t_1}\, c(t_1)\left|\phi(a^*) - \omega'_{n+1}\right| \geq \min_{\alpha=0,1} \xi^*_{t_\alpha} c(t_\alpha) = \frac{c(t_0)c(t_1)}{c(t_0) + c(t_1)} = v,$$

and since this is true for any $\phi \in \Phi$, x^* is an optimal strategy.

Remark: The results of this section remain true even if the time when Predictor is required to make a prediction is allowed to depend on the observed sequence in some predetermined fashion. For instance, we may ask Predictor to predict the next digit as soon as he observes a certain number k of ones or as soon as the number of observed digits reaches some n, whichever occurs first.

The space Φ of Predictor's pure strategies is then the set of all mappings ϕ from a finite set B into $\{0,1\}$, where the set B is a finite collection of finite strings of zeros and ones with the property that for every binary sequence $\omega \in \Omega$ there is a string $b \in B$, which is the initial segment of ω.

The set B is determined by the requirement imposed upon the prediction time. For instance, in the case mentioned above the set B will consist of all strings of at most n zeros and ones which contain at most k zeros.

The only modification needed in order to extend the results of this section to this more general case is to replace the set A^n by the set B.

Example: Consider the game $G(c(t),n)$ with $n = 5$ and

$$c(t) = \begin{cases} \frac{1}{5}(6-t) & \text{if } t = 1,\ldots,6, \\ 0 & \text{if } t > 6. \end{cases}$$

To solve the game we first compute $\lambda(a_i\alpha)$ for each $a_i \in A^5$, $\alpha = 0,1$. These are in the first two columns of Table 2. Then compute $c(\lambda(a_i\alpha))$ for (19) and $h(a_i)$ from (17) - see Table 2.

Hence the value v of the game $G(c(t),n)$ is $v = 1.5$ and Predictor's optimal strategy f^* as computed from (16) is in the last column of Table 2. Emitter's optimal strategies are computed from Proposition 4.3. We obtain

$$\xi_t^* = \begin{cases} .25 & \text{if } t = 3, \\ .75 & \text{if } t = 5, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu_3^*(\{(011)^\infty,(100)^\infty\}) = 1, \quad \mu_5^*(\{(01101)^\infty,(10010)^\infty\}) = 1$$

and arbitrary otherwise. In other words, Emitter should produce the sequence 011011011... or 110110110... with probability 1/4 and the sequence 0110101101... or 1101011010... with probability 3/4.

5. NONZERO-SUM GAMES.

In the previous two sections we dealt exclusively with zero-sum games. That is, the payoff to Predictor was always assumed to be the negative amount of the payoff to Emitter.

Let us now modify the game $G(c(t),n)$ of the previous section by adding to its description the assumption:

(4) The payoff to Predictor is one if his prediction is a correct one and zero otherwise.

We obtain a new nonzero-sum game

$$G'(c(t),n) = (S,\Phi,W_1,W_2),$$

where the strategy spaces S,Φ and Emitter's payoff function W_1 are same as in the zero-sum game $(G(c(t),n)$. The Predictor's payoff function W_2 is defined by

$$W_2(s,\phi) = \begin{cases} 1 & \text{if } \phi(\underline{\omega}|^n) = \omega_{n+1}, \\ 0 & \text{if } \phi(\underline{\omega}|^n) \neq \omega_{n+1}, \end{cases}$$

where ω is the sequence s_t.

Proposition 5.1: The nonzero-sum game $G'(c(t),n)$ has an equilibrium point (x^*,f^*), where f^* is defined by (16) and x^* is defined as in Proposition 4.3 with the exception that $\xi^*_{t_0} = \xi^*_{t_1} = 1/2$. Furthermore, all equilibrium points are equivalent and interchangeable

yielding the minimax payoffs v and $1/2$ to Emitter and Predictor respectively.

Proof: Let a^* be defined by (18). Then for all $f \in F$ by the definition of x^*

$$W_2(x^*,f) = \xi^*_{t_0} f_0(a^*) + \xi^*_{t_1} f_1(a^*) = \frac{1}{2},$$

so that trivially

$$\sup_{f\in F} W_2(x^*,f) = W_2(x^*,f^*) = \frac{1}{2}.$$

Since f^* is a minimax strategy in $G(c(t),n)$

$$W_1(s,f^*) \leq v$$

for any $s \in S$.

Now by the definition of x^* and f^*

$$W_1(x^*,f^*) = c(t_0)\xi^*_{t_0} f^*_1(a^*) + c(t_1)\xi^*_{t_1} f^*_0(a^*)$$

$$= c(t_0)c(t_1)[c(t_0) + c(t_1)]^{-1} = h(a^*) = v.$$

Hence

$$\sup_{s\in S} W_1(s,f^*) = W_1(x^*,f^*) = v,$$

which together with (20) proves that (x^*,f^*) is an equilibrium point yielding minimax payoffs.

Finally it is easy to see that no equilibrium point can yield payoff greater than minimax for either of the players since this would decrease the other player's payoff below his minimax value. Interchangebility is obvious.

The last type of an Emission-Prediction game considered in this paper is the most general one and is obtained by further relaxing the restrictions on Predictor.

(1) Emitter chooses a period $t = 1,2,\ldots$ and a periodic binary sequence with this period.

(2) Predictor is allowed to observe as many digits of the sequence as he wishes and after he decides to cease observation he is required to predict the next digit.

(3) The payoff to Emitter is zero if the prediction is correct and is equal to $c_1(t) \geq 0$ if the prediction is wrong and the sequence has period t.

(4) The payoff to Predictor is zero if his prediction is wrong and is equal to $c_2(n) \geq 0$ if it is correct and the length of the observed segment was n.

The constants $c_1(t)$ and $c_2(n)$ represent penalties for long periods and lengthy observation. We assume that

$$t = 1,2,\ldots \Rightarrow 1 = c_1(1) \geq c_1(t) \geq c_1(t+1), \quad \lim_{t\to\infty} c_1(t) = 0$$

and

$$n = 0,1,\ldots \Rightarrow 1 = c_2(0) \geq c_2(n) \geq c_2(n+1), \quad \lim_{n\to\infty} c_2(n) = 0.$$

We will denote the resulting nonzero-sum game by $G(c_1(t), c_2(n))$.

The space S of Emitter's pure strategies in this game is same as before. Predictor's pure strategies $\psi \in \Psi$ are pairs

$$\psi = (B,\phi),$$

where B is a finite set of finite strings of zeros and ones such that for every $\omega \in \Omega$ there is some $b \in B$ which is an initial segment of ω. ϕ is a mapping from B into $\{0,1\}$. Clearly, the set Ψ of all Predictor's pure strategies is now countably infinite.

Let $s = (t,s_t)$, $\psi = (B,\phi)$, $\omega = s_t$, $b \in B$ be the initial segment of ω, and let n be the length of b. The payoffs are now given by:

$$W_1(s,\psi) = \begin{cases} c_1(t) & \text{if } \phi(b) \neq \omega_{n+1}, \\ 0 & \text{if } \phi(b) = \omega_{n+1}, \end{cases}$$

$$W_2(s,\psi) = \begin{cases} c_2(n) & \text{if } \phi(b) = \omega_{n+1}, \\ 0 & \text{if } \phi(b) \neq \omega_{n+1}. \end{cases}$$

Proposition 5.2: The game $G(c_1(t),c_2(n))$ has an equilibrium point.

Proof: Let for $j = 1,2$

$$\rho_{j,1}(s,s') = \sup_{\psi\in\Psi}|W_j(s,\psi) - W_j(s',\psi)| \quad \text{and}$$

$$\rho_{j,2}(\psi,\psi') = \sup_{s\in S}|W_j(s,\psi) = W_j(s,\psi')|$$

be Wald's (pseudo) metrics [8] on S and ψ respectively. Let $\varepsilon > 0$ and

$$S_\varepsilon = \{(t,s_t) \in S : c_1(t) \geq \frac{\varepsilon}{2}\},$$

$$\psi_\varepsilon = \{(B,\phi) \in \psi : c_2(|B|) \geq \frac{\varepsilon}{2}\},$$

where $|B|$ is the length of the longest string $b \in B$. Since $\lim_{t\to\infty} c_1(t) = \lim_{n\to\infty} c_2(n) = 0$ the finite sets S_ε and ψ_ε are ε-nets in the (pseudo) metric spaces (S,ρ_{11}) and (ψ,ρ_{22}) respectively and hence these spaces are totally bounded ("conditionally compact" in Wald's terminology). By [8], Theorem 2.1, so are then the spaces (S,ρ_{12}) and (ψ,ρ_{21}). Let, for $k = 1,2,\ldots$, (x^*_k,y^*_k) be an equilibrium point in the finite game $(S_{1/k},\psi_{1/k},W_1,W_2)$. Then there is a subsequence $(x^*_{k_i},y^*_{k_i})$, $i = 1,2,\ldots$, and a pair of mixed strategies $x^* \in X$, $y^* \in Y$ such that as $i \to \infty$, $j = 1,2$, $\rho_{j,1}(x^*_{k_i},x^*) \to 0$ and $\rho_{j,2}(y^*_{k_i},y^*) \to 0$. In particular this implies, as $i \to \infty$,

$$|W_1(x^*_{k_i},y^*_{k_i}) - W_1(x^*,y^*)| \to 0, \tag{21}$$

$$|\sup_{x\in X_{1/k_i}} W_1(x,y^*) - \sup_{x\in X} W_1(x,y^*)| \to 0, \tag{22}$$

$$|\sup_{x\in X_{1/k_i}} W_1(s,y^*_{k_i}) - \sup_{x\in X} W_1(x,y^*)| \to 0. \tag{23}$$

Finally, since $(x^*_{k_i},y^*_{k_i})$ is an equilibrium point in G_{k_i}

$$\sup_{x\in X_{1/k_i}} W_1(x,y^*_{k_i}) = W_1(x^*_{k_i},y^*_{k_i})$$

whence letting $i \to \infty$ in (21) through (23) we obtain

$$\sup_{x\in X} W_1(x,y^*) = W_1(x^*,y^*).$$

Similarly,

$$\sup_{y\in Y} W_2(x^*,y) = W_2(x^*,y^*),$$

so that (x^*,y^*) is an equilibrium point in $G(c_1(t),c_2(n))$ and the proposition is proved.

Remark: We suspect that much stronger statements, namely that all equilibrium points are equivalent and interchangeable, can be proved, but we have not been able to prove this.

i	a_i	$I_t(a1)$ t 1 2 3 4 5	$I_t(a0)$ t 1 2 3 4 5	$f^*(a) = (f_0^*(a), f_1^*(a))$ Case 1	Case 2	Case 3	Case 4
1	0 0 0 0 0		1				
2	0 0 0 0 1		1				
3	0 0 0 1 0		1 1	(1,0)			(1,0)
4	0 0 0 1 1		1				
5	0 0 1 0 0	1	1 1	(0,1)	(1,0)		$(\frac{1}{2},\frac{1}{2})$
6	0 0 1 0 1		1				
7	0 0 1 1 0		1 1	(1,0)			(1,0)
8	0 0 1 1 1		1			(1,0)	
9	0 1 0 0 0	1	1	arbitrary	(0,1)		$(\frac{1}{2},\frac{1}{2})$
10	0 1 0 0 1		1 1	(1,0)	(1,0)		(1,0)
11	0 1 0 1 0	1	1	(0,1)	(0,1)		$(\frac{1}{2},\frac{1}{2})$
12	0 1 0 1 1		1	(1,0)	(1,0)		(1,0)
13	0 1 1 0 0	1	1	arbitrary	(0,1)		$(\frac{1}{2},\frac{1}{2})$
14	0 1 1 0 1	1	1	(1,0)	(1,0)		(1,0)
15	0 1 1 1 0	1	1	arbitrary	(0,1)		$(\frac{1}{2},\frac{1}{2})$
16	0 1 1 1 1		1	(1,0)	(1,0)		(1,0)
17	1 0 0 0 0	1		(0,1)	(0,1)		(0,1)
18	1 0 0 0 1	1	1	arbitrary	(1,0)		$(\frac{1}{2},\frac{1}{2})$
19	1 0 0 1 0	1	1	(0,1)	(0,1)		(0,1)
20	1 0 0 1 1	1	1	arbitrary	(1,0)		$(\frac{1}{2},\frac{1}{2})$
21	1 0 1 0 0	1		(0,1)	(0,1)		(0,1)
22	1 0 1 0 1	1	1	(1,0)	(1,0)		$(\frac{1}{2},\frac{1}{2})$
23	1 0 1 1 0	1 1		(0,1)	(0,1)		(0,1)
24	1 0 1 1 1	1	1	arbitrary	(1,0)		$(\frac{1}{2},\frac{1}{2})$
25	1 1 0 0 0	1				(0,1)	
26	1 1 0 0 1	1 1		(0,1)			(0,1)
27	1 1 0 1 0	1					
28	1 1 0 1 1	1 1	1	(1,0)			$(\frac{1}{2},\frac{1}{2})$
29	1 1 1 0 0	1			(0,1)		
30	1 1 1 0 1	1 1		(0,1)			(0,1)
31	1 1 1 1 0	1					
32	1 1 1 1 1	1					

TABLE 1.

Row i of Table 1	$\lambda(a_i 0)$	$\lambda(a_i 1)$	$c(\lambda(a_i 0))$	$c(\lambda(a_i 1))$	$h(a_i)$	$(f_0^*(a_i), f_1^*(a_i))$
1	1	6	1	0	0	(1,0)
22	2	5	.8	.2	.16	(.8,.2)
10	3	6	.6	0	0	(1,0)
19	3	5	.6	.2	1.5	(.75,.25)
28	3	4	.6	.4	.24	(.6,.4)
3,7,18,20,24	4	6	.4	0	0	(1,0)
5	4	3	.4	.6	.24	(.4,.6)
2,4,6,8,12,16	5	6	.2	0	0	(1,0)
9,3,15	5	4	.2	.4	.133	(.333,.666)
14	5	3	.2	.6	1.5	(.25,.75)
11	5	2	.2	.8	.16	(.2,.8)
17,21,25,27,29,31	6	5	0	.2	0	(0,1)
26,28,30	6	4	0	.4	0	(0,1)
23	6	3	0	.6	0	(0,1)
32	6	1	0	1	0	(0,1)

TABLE 2.

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13. ABSTRACT

In this paper several infinite two-person games are studied, all having the following common structure: Player 1 (Emitter) produces a binary periodic sequence. Player 2 (Predictor) observes some initial segment of this sequence and then tries to predict the next digit. The payoff to Emitter is zero if the prediction is a correct one. The games differ in additional assumptions--those are in particular: (1) Predictor required to make his prediction after observing a prescribed number of digits of the sequence. (2) Predictor allowed to observe any number of digits but earning a decreasing amount for each correct prediction as the number increases. (3) The period of the emitted sequence being chosen by random from some fixed distribution. (4) Emitter allowed to choose the period but being paid a decreasing amount for incorrect prediction as the period increases. Combining these assumptions two zero-sum and two nonzero-sum games are obtained. It is shown that all these games possess a solution, some are at least partially solved and their further properties investigated.

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